University of Cambridge
Department of
Materials Science & Metallurgy

Introduction to the Finite Element Method (FEM)

Lecture 3
First Order Two Dimensional Shape Functions

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The exact solution to the function $\phi(X)$ has been approximated with a finite element in Fig.1. It has been approximated in this instance with both a first and second order element. We derived the shape functions for one dimensional first and second order elements earlier in the course in order to predict ϕ at positions other than the nodes.

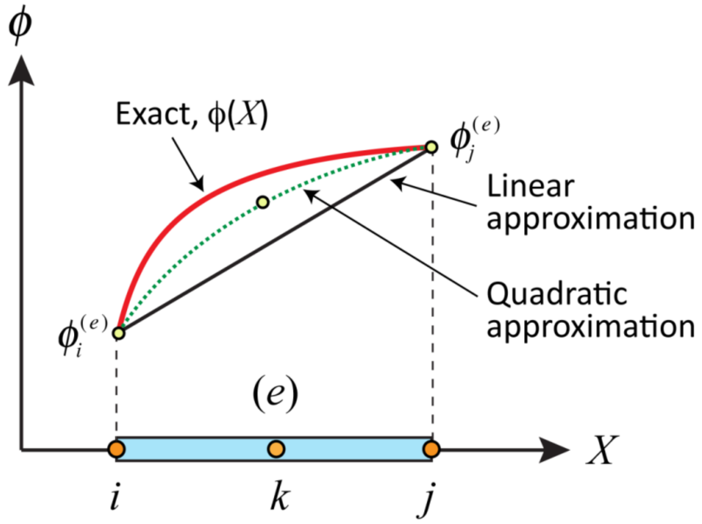


Figure 1: First and second order, one dimensional finite element discretisation of $\phi(X)$

However, most problems of practical interest present themselves in 2 or 3 dimensions. An example is given in Fig.2, in which $\phi=f(x,y)$. The exact solution in this instance is approximated with a two dimensional, first order (bi-linear) triangular element.

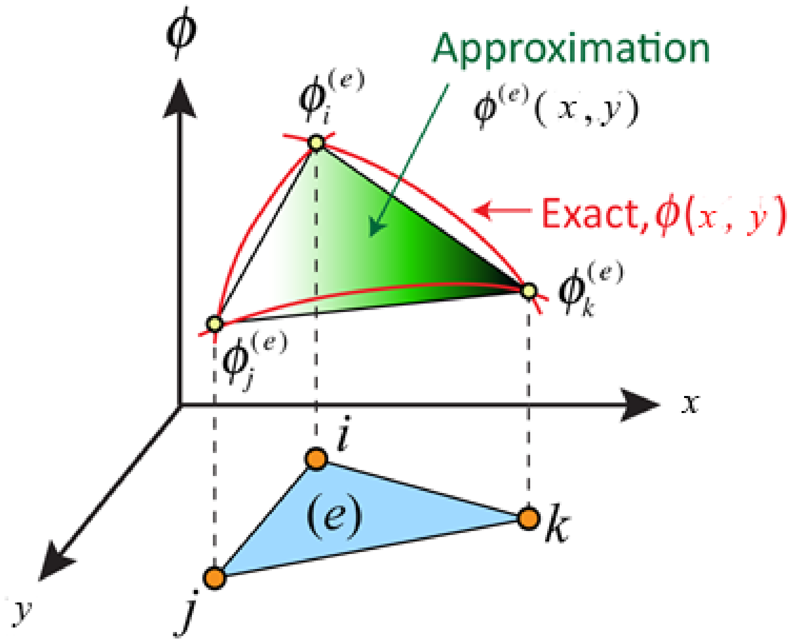


Figure 2: First order, two dimensional finite element discretisation of $\phi(x,y)$

First Order Two Dimensional Shape Functions in Global Coordinates

For the element shown in Fig.2:

$$\emptyset(x,y) = \sum_{i=1}^{n} S_i(x,y)\phi_i \quad (1)$$

$$\emptyset(x,y) = S_1\phi_1 + S_2\phi_2 + S_3\phi_3 \text{ for } i = 1,2,3 \quad (2)$$

For a 2-dimensional first order element the bi-linear interpolation function S is:

$$S_i = a_i + b_i x + c_i y \quad (3)$$

Therefore:

$$S_1 = a_1 + b_1 x + c_1 y \quad (4)$$

$$S_2 = a_2 + b_2 x + c_2 y \quad (5)$$

$$S_3 = a_3 + b_3 x + c_3 y \quad (6)$$

We need to find therefore the values of a_i, b_i and c_i in terms of the known x_i and y_i.

Recall that the shape function is equal to one at its node and zero at the other nodes such that, ***at node 1***:

$$S_1(x_1,y_1) = a_1 + b_1x_1 + c_1y_1 = 1 \quad (7)$$

$$S_1(x_2,y_2) = a_1 + b_1x_2 + c_1y_2 = 0 \quad (8)$$

$$S_1(x_3,y_3) = a_1 + b_1x_3 + c_1y_3 = 0 \quad (9)$$

Solving the simultaneous equations yields values for a_1, b_1 and c_1:

$$a_1 = \frac{x_2y_3 - x_3y_2}{2A} \quad (10)$$

$$b_1 = \frac{y_2 - y_3}{2A} \quad (11)$$

$$c_1 = \frac{x_3 - x_2}{2A} \quad (12)$$

where *A* is the element area. More generally:

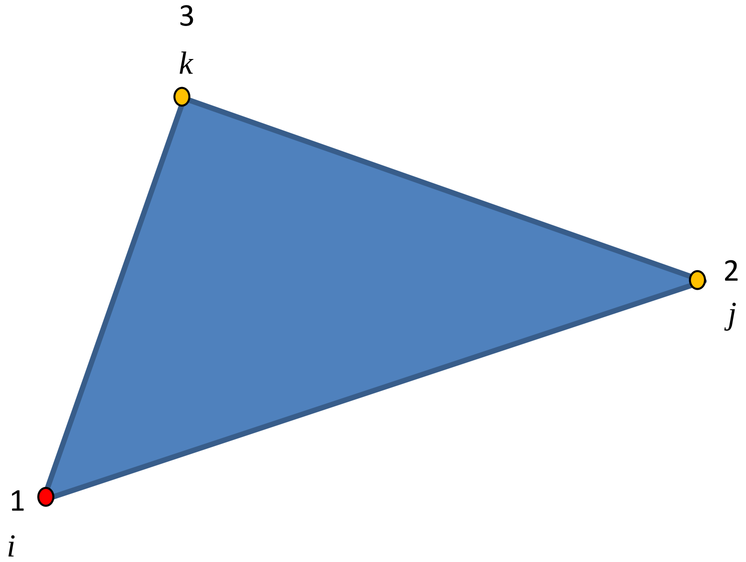
$$a_i = \frac{x_jy_k - x_ky_j}{2A} \quad (13)$$

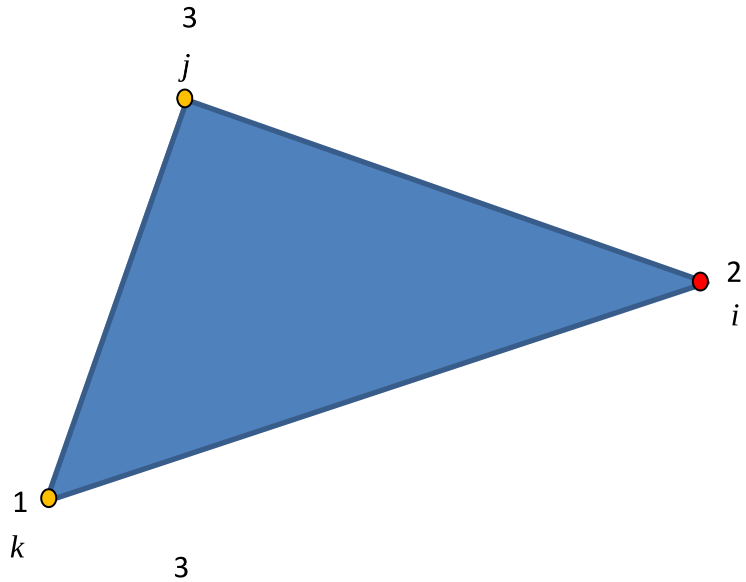
$$b_i = \frac{y_j - y_k}{2A} \quad (14)$$

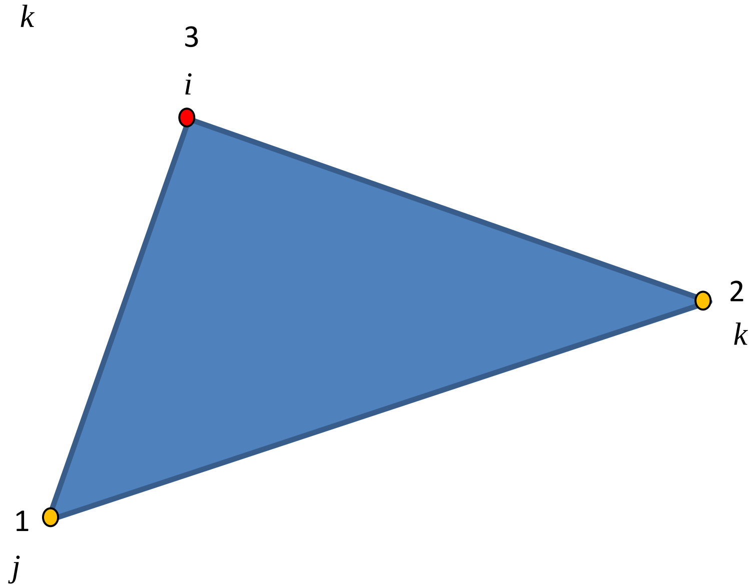
$$c_i = \frac{x_k - x_j}{2A} \quad (15)$$

Which we can use to find a_2, b_2, c_2 and a_3, b_3, c_3 using a cyclic permutation of i, j and k – i.e:

Starting Node ↓	*i,j,k* Notation		
	i	*j*	*k*
i	*1*	*2*	*3*
j	2	3	1
k	3	1	2







Therefore, if

$$a_i = \frac{x_j y_k - x_k y_j}{2A} \tag{16}$$

$$b_i = \frac{y_j - y_k}{2A} \tag{17}$$

$$c_i = \frac{x_k - x_j}{2A} \tag{18}$$

then

$$a_2 = \frac{x_3 y_1 - x_1 y_3}{2A} \tag{19}$$

$$b_2 = \frac{y_3 - y_1}{2A} \tag{20}$$

$$c_2 = \frac{x_1 - x_3}{2A} \tag{21}$$

and

$$a_3 = \frac{x_1 y_2 - x_2 y_1}{2A} \tag{22}$$

$$b_3 = \frac{y_1 - y_2}{2A} \tag{23}$$

$$c_3 = \frac{x_2 - x_1}{2A} \tag{24}$$

To give:

$$S_1 = \frac{x_2 y_3 - x_3 y_2}{2A} + \left(\frac{y_2 - y_3}{2A}\right) x + \left(\frac{x_3 - x_2}{2A}\right) y \tag{25}$$

$$S_2 = \frac{x_3 y_1 - x_1 y_3}{2A} + \left(\frac{y_3 - y_1}{2A}\right) x + \left(\frac{x_1 - x_3}{2A}\right) y \tag{26}$$

$$S_3 = \frac{x_1 y_2 - x_2 y_1}{2A} + \left(\frac{y_1 - y_2}{2A}\right) x + \left(\frac{x_2 - x_1}{2A}\right) y \tag{27}$$

So that, using Eqn.2:

$$\emptyset(x, y) = \left(\frac{x_2 y_3 - x_3 y_2}{2A} + \left(\frac{y_2 - y_3}{2A}\right) x + \left(\frac{x_3 - x_2}{2A}\right) y\right) \phi_1 + \left(\frac{x_3 y_1 - x_1 y_3}{2A} + \left(\frac{y_3 - y_1}{2A}\right) x + \left(\frac{x_1 - x_3}{2A}\right) y\right) \phi_2 + \left(\frac{x_1 y_2 - x_2 y_1}{2A} + \left(\frac{y_1 - y_2}{2A}\right) x + \left(\frac{x_2 - x_1}{2A}\right) y\right) \phi_3 \tag{28}$$

The area of a triangular element is given by:

$$A = \frac{1}{2} \det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \frac{1}{2} [(x_2 y_3 - x_3 y_2) + x_1 (y_2 - y_3) + y_1 (x_3 - x_2)] \tag{29}$$

Using Eqn.28 and Eqn.29, find $\emptyset^{(e)}(3.3,2.6)$

i		*j*		*k*	
x_1	1.5	x_2	1.3	x_3	4.2
y_1	2.0	y_2	5.0	y_3	4.0
ϕ_1	*15.0*	ϕ_2	*12.0*	ϕ_3	11.0