



University of Cambridge
Department of
Materials Science & Metallurgy

Introduction to the Finite Element Method (FEM)

Lecture 2 First and Second Order One Dimensional Shape Functions

Dr. J. Dean



Discretisation

Consider the temperature distribution along the one-dimensional fin in Fig.1.

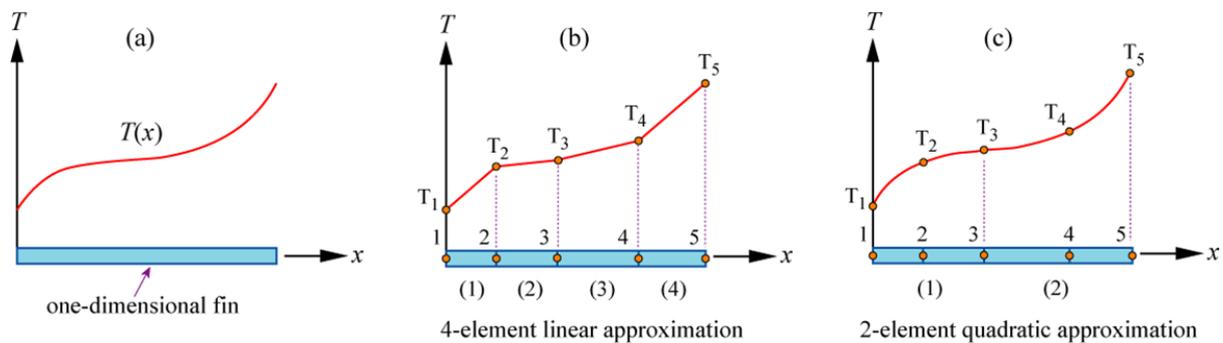


Figure 1: Depiction of a piecewise approximation to a continuous function

A one-dimensional continuous temperature distribution with an infinite number of unknowns is shown in (a). The fin is discretised in (b) – i.e. divided into 4 subdomains (or elements). The nodes are numbered consecutively from left to right, as are the elements. The elements are *first order* elements; the interpolation scheme between the nodes is therefore linear. Note that there are only 5 nodes for this system, since the internal nodes are shared between the elements. Since we are only solving for temperature, there are only 5 degrees of freedom in this model of the continuous system. It should be clear that a better approximation for $T(x)$ would be obtained if the number of elements was increased (i.e. if the element lengths were reduced). It is also apparent that the nodes should be placed closer together in regions where the temperature (or any other unknown solution) changes rapidly. It is useful also to place a node wherever a step change in temperature is expected and where a numerical value of the temperature is needed. It is good practice to continue to increase the number of nodes until a converged solution is reached.

In (c), the fin has been divided into two subdomains – elements 1 and 2. However, in this instance we have chosen to use a *second order* (quadratic) element. These elements contain ‘midside’ nodes as shown, and the interpolation between the nodes is quadratic which permits a much closer approximation to the real system. For this model system there are still just 5 degrees of freedom. However, the analysis takes longer for (c) than it does for (b) because the quadratic interpolation (which calculates the temperature at locations between the nodes) is more demanding than the corresponding linear case.¹

(There is often a trade-off between a high number of first order elements requiring little computation and a smaller number of second order elements requiring more heavy computation to be made, which affects both the analysis time and the solution accuracy. The choice depends to a large extent on the problem being solved.)

¹ For a system with only 5 degrees of freedom the difference would be imperceptible using modern desktop machines. This would not be the case for large systems containing hundreds of thousands of degrees of freedom.

Shape Functions

We can use (for instance) the direct stiffness method to compute degrees of freedom at the element nodes. However, we are also interested in the value of the solution at positions inside the element. To calculate values at positions other than the nodes we interpolate between the nodes using shape functions.

A one-dimensional element with length L is shown in Fig.2. It has two nodes, one at each end, denoted i and j , and known nodal temperatures T_i and T_j . We can deduce automatically that the element is first order (linear) since it contains no 'midside' nodes.

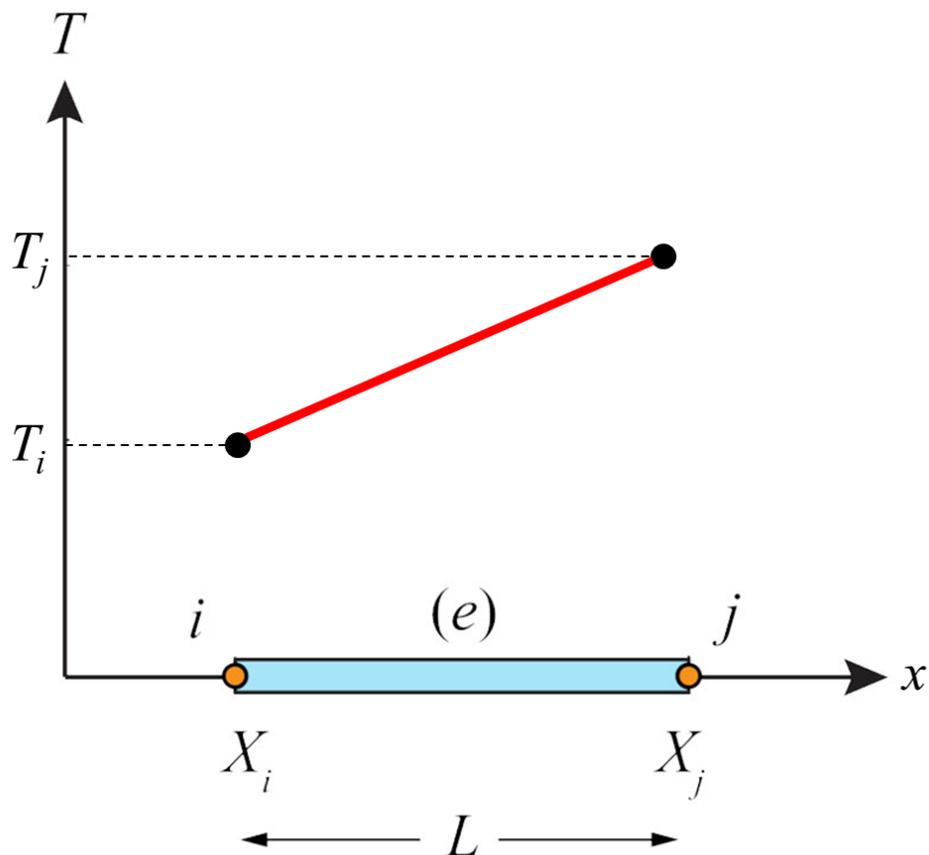


Figure 2: One dimensional linear element with temperature degrees of freedom

We need to derive a function to compute values of the temperature at locations between the nodes. This interpolation function is called the shape function. We demonstrate its derivation for a 1-dimensional linear element here. Note that, for linear elements, the polynomial interpolation function is first order. If the element was second order, the polynomial function would be second order (quadratic), and so on.

Since the element is first order, the temperature varies linearly between the nodes and the equation for T is:

$$T(x) = a + bx \tag{1}$$

We can therefore write:

$$T_i = a + bX_i \quad (2)$$

$$T_j = a + bX_j \quad (3)$$

which are simultaneous. To determine the coefficients a and b :

$$\frac{T_i - a}{X_i} = b \quad (4)$$

$$\frac{T_j - a}{X_j} = b \quad (5)$$

$$\frac{T_i - a}{X_i} = \frac{T_j - a}{X_j} \quad (6)$$

$$(T_i - a)X_j = (T_j - a)X_i \quad (7)$$

$$T_iX_j - aX_j = T_jX_i - aX_i \quad (8)$$

$$T_iX_j - T_jX_i = a(X_j - X_i) \quad (9)$$

$$\frac{T_iX_j - T_jX_i}{(X_j - X_i)} = a \quad (10)$$

$$\frac{T_iX_j - T_jX_i}{L} = a \quad (11)$$

and

$$T_i - bX_i = a \quad (12)$$

$$T_j - bX_j = a \quad (13)$$

$$T_i - bX_i = T_j - bX_j \quad (14)$$

$$bX_j - bX_i = T_j - T_i \quad (15)$$

$$b(X_j - X_i) = T_j - T_i \quad (16)$$

$$b = \frac{T_j - T_i}{(X_j - X_i)} \quad (17)$$

$$b = \frac{T_j - T_i}{L} \quad (18)$$

Substitution of Eqns.11 and 18 into Eqn.1 yields:

$$T(x) = \frac{T_iX_j - T_jX_i}{L} + \left(\frac{T_j - T_i}{L}\right)x \quad (19)$$

$$T(x) = \frac{T_iX_j - T_jX_i}{L} + \frac{T_jx - T_ix}{L} \quad (20)$$

$$T(x) = \frac{T_i X_j}{L} - \frac{T_j X_i}{L} + \frac{T_j x}{L} - \frac{T_i x}{L} \quad (21)$$

$$T(x) = \frac{T_i X_j}{L} - \frac{T_i x}{L} + \frac{T_j x}{L} - \frac{T_j X_i}{L} \quad (22)$$

$$T(x) = T_i \left(\frac{X_j - x}{L} \right) + T_j \left(\frac{x - X_i}{L} \right) \quad (23)$$

It should be clear from Eqn.23 that the nodal temperature values are multiplied by linear functions of x – the shape functions. The functions are denoted by S with a subscript to indicate the node with which a specific shape function is associated. In the case presented:

$$S_i = \left(\frac{X_j - x}{L} \right) \quad \text{Draw linear shape functions} \quad (24)$$

$$S_j = \left(\frac{x - X_i}{L} \right) \quad (25)$$

And Eqn.23 becomes

$$T(x) = S_i T_i + S_j T_j \quad (26)$$

In matrix form

$$T_x^e = [S_i \quad S_j] \begin{Bmatrix} T_i \\ T_j \end{Bmatrix} \quad (27)$$

For the case shown in Fig.3, calculate T at $x = 3.3$

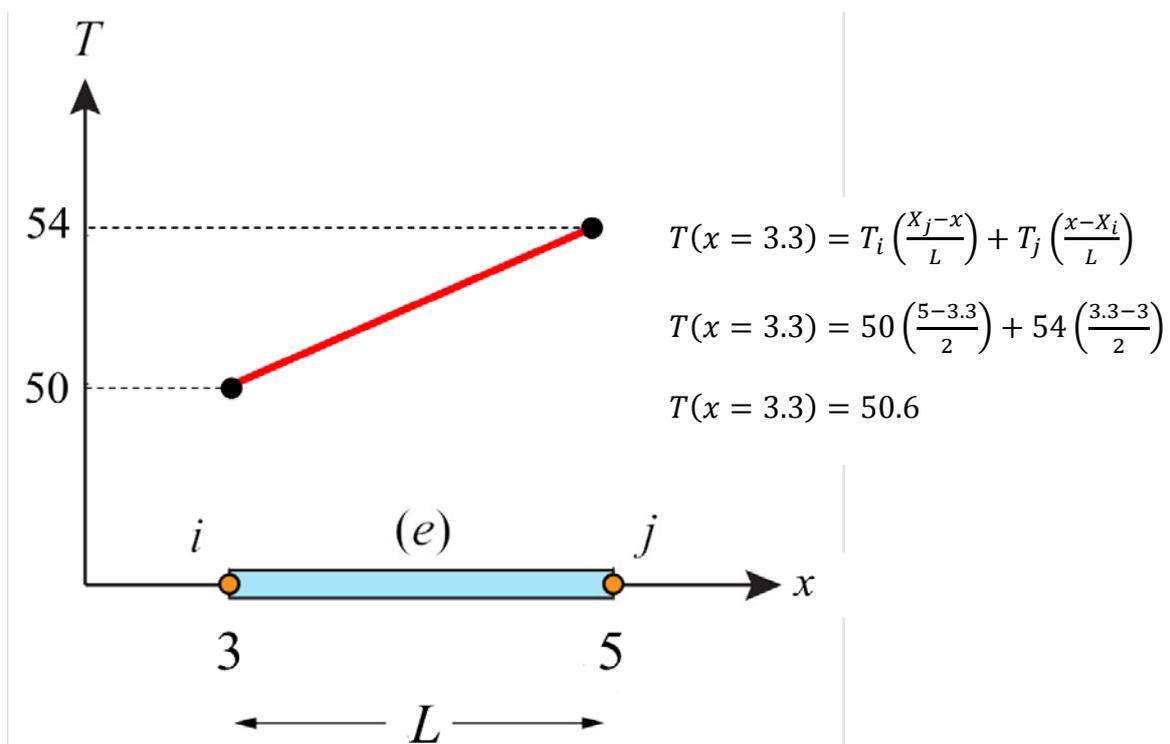
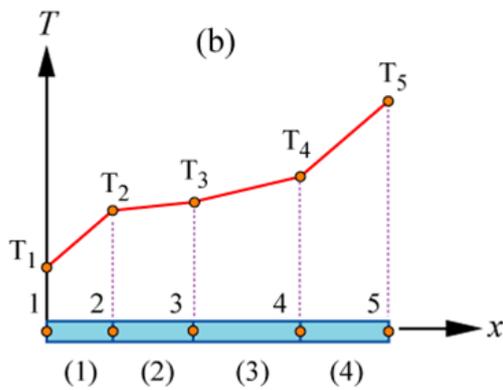


Figure 3: 1-dimensional linear element with known nodal temperatures and positions

From inspection of Eqn.26 we can deduce that each shape function has a value of 1 at its own node and a value of zero at the other nodes. The sum of the shape functions sums to one. The shape functions are also first order, just as the original polynomial was. The shape functions would have been quadratic if the original polynomial has been quadratic.

A continuous, piecewise smooth equation for the one dimensional fin first shown in Fig.1 can be constructed by connecting the linear element equations. We know that the temperature at any point in any element can be found from the nodal temperatures T_i and the shape functions, S_i . For the following system:

$$T_x^e = S_i T_i + S_j T_j \quad X_i \leq x \leq X_j \quad (28)$$



Element #	Node	
	<i>i</i>	<i>j</i>
1	1	2
2	2	3
3	3	4
4	4	5

4-element linear approximation

$$T_x^1 = S_1^1 T_1 + S_2^1 T_2 \quad S_1^1 = \frac{X_2 - x}{X_2 - X_1} \quad S_2^1 = \frac{x - X_1}{X_2 - X_1} \quad X_1 \leq x \leq X_2 \quad (29)$$

$$T_x^2 = S_2^2 T_2 + S_3^2 T_3 \quad S_2^2 = \frac{X_3 - x}{X_3 - X_2} \quad S_3^2 = \frac{x - X_2}{X_3 - X_2} \quad X_2 \leq x \leq X_3 \quad (30)$$

$$T_x^3 = S_3^3 T_3 + S_4^3 T_4 \quad S_3^3 = \frac{X_4 - x}{X_4 - X_3} \quad S_4^3 = \frac{x - X_3}{X_4 - X_3} \quad X_3 \leq x \leq X_4 \quad (31)$$

$$T_x^4 = S_4^4 T_4 + S_5^4 T_5 \quad S_4^4 = \frac{X_5 - x}{X_5 - X_4} \quad S_5^4 = \frac{x - X_4}{X_5 - X_4} \quad X_4 \leq x \leq X_5 \quad (32)$$

$$S_2^1 \neq S_2^2, S_3^2 \neq S_3^3 \text{ and } S_4^3 \neq S_4^4$$

The temperature gradient through the an individual element $\frac{dT^e}{dx}$ can be found from the derivative of Eqn.33

$$T_x^e = T_i \left(\frac{X_j - x}{L} \right) + T_j \left(\frac{x - X_i}{L} \right) \quad (33)$$

$$T_x^e = \frac{X_j T_i}{L} - \frac{x T_i}{L} + \frac{x T_j}{L} + \frac{X_i T_j}{L} \quad (34)$$

$$T_x^e = \frac{x T_j}{L} - \frac{x T_i}{L} + \frac{X_j T_i}{L} - \frac{X_i T_j}{L} \quad (35)$$

$$T_x^e = \frac{x}{L} (T_j - T_i) + \frac{1}{L} (X_j T_i - X_i T_j) \quad (36)$$

such that

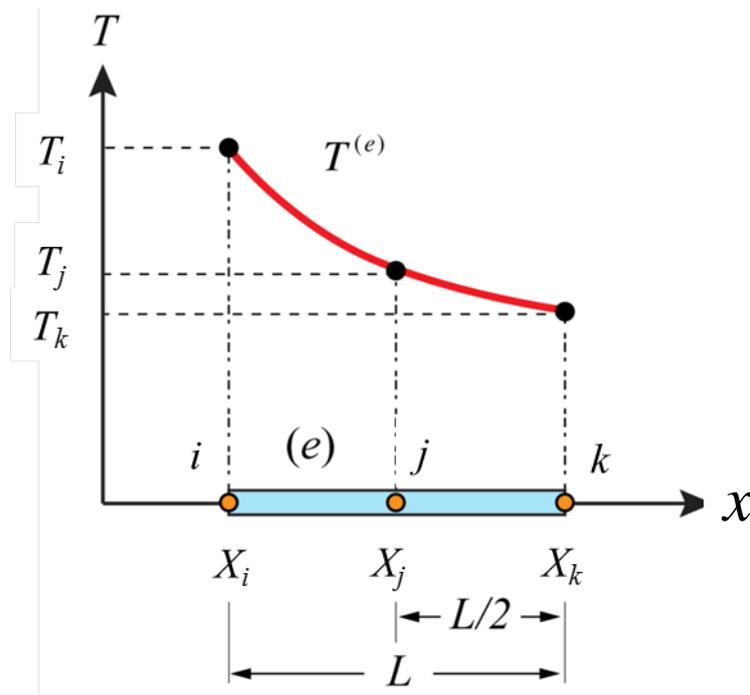
$$\frac{dT_x^e}{dx} = \frac{T_j - T_i}{L} \quad (37)$$

We can check that our shape functions are correct by knowing that the shape function derivatives sum to zero.

1-Dimensional Quadratic Elements

A one-dimensional quadratic element is shown in Fig.4. We can deduce immediately that the element order is greater than one because the interpolation between the nodes is non-linear. We can determine from inspection that the element is quadratic (second order) because there's a 'midside' node. We know therefore that the function approximating the solution is a second order polynomial:

$$T_x^e = a + bx + cx^2 \quad (38)$$



The shape functions S_i can be determined by solving Eqn.38 using known T_i at known X_i to give:

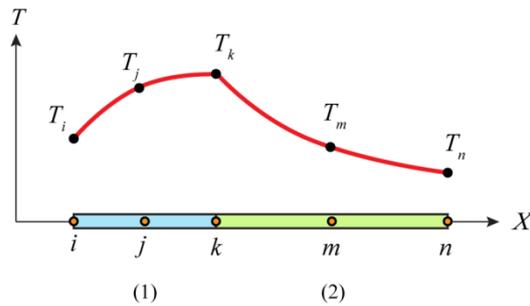
$$T_x^e = S_i T_i + S_j T_j + S_k T_k \quad (39)$$

$$S_i = \frac{2}{L^2} (x - X_k)(x - X_j) \quad (40)$$

$$S_j = \frac{-4}{L^2} (x - X_i)(x - X_k) \quad (41)$$

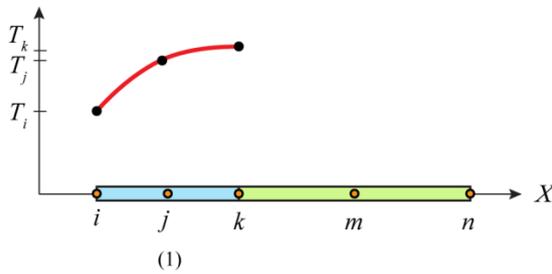
$$S_k = \frac{2}{L^2} (x - X_i)(x - X_j) \quad (42)$$

Using the quadratic shape functions for a single element (Eqns.40-42), we can assemble a corresponding set of equations for a larger system:

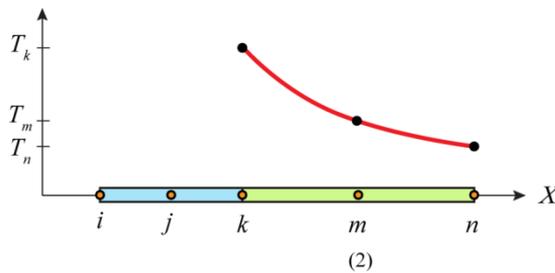


Nodes			
Element #	<i>i</i>	<i>j</i>	<i>k</i>
1	<i>i</i>	<i>j</i>	<i>k</i>

Nodes			
Element #	<i>i</i>	<i>j</i>	<i>k</i>
2	<i>k</i>	<i>m</i>	<i>n</i>



$$T_x^1 = [S_i^1 \quad S_j^1 \quad S_k^1] \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$



$$T_x^2 = [S_k^2 \quad S_m^2 \quad S_n^2] \begin{Bmatrix} T_k \\ T_m \\ T_n \end{Bmatrix}$$

$$T_x^1 = S_i^1 T_i + S_j^1 T_j + S_k^1 T_k \tag{43}$$

$$S_i^1 = \frac{2}{L^2} (x - X_k)(x - X_j)$$

$$S_j^1 = \frac{-4}{L^2} (x - X_i)(x - X_k)$$

$$S_k^1 = \frac{2}{L^2} (x - X_i)(x - X_j)$$

$$T_x^2 = S_k^2 T_k + S_m^2 T_m + S_n^2 T_n \tag{44}$$

$$S_k^2 = \frac{2}{L^2} (x - X_n)(x - X_m)$$

$$S_m^2 = \frac{-4}{L^2} (x - X_k)(x - X_n)$$

$$S_n^2 = \frac{2}{L^2} (x - X_k)(x - X_m)$$

The element shape functions are stored within the element in commercial FE codes. The positions X_i are generated (and stored) when the mesh is created. Once the nodal degrees of freedom are known, the solution at any point between the nodes can be calculated using the (stored) element shape functions and the (known) nodal positions. The nodal degrees of freedom are calculated using a method such as that shown previously in *lecture 1*.

For the cases presented above, simple 1-dimensional elements were most appropriate, but for many practical applications we may encounter more complex 2- and 3-dimensional geometry. A suitable set of element types are shown in Fig.4 encompassing a range of geometry and element order. A meshed geometry is shown in Fig.5.

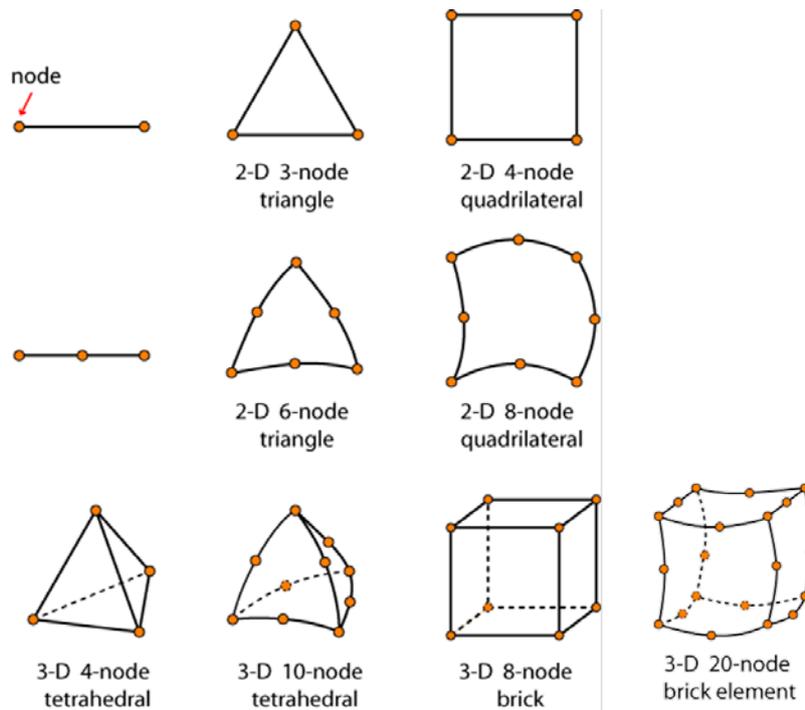


Figure 4: Example element types

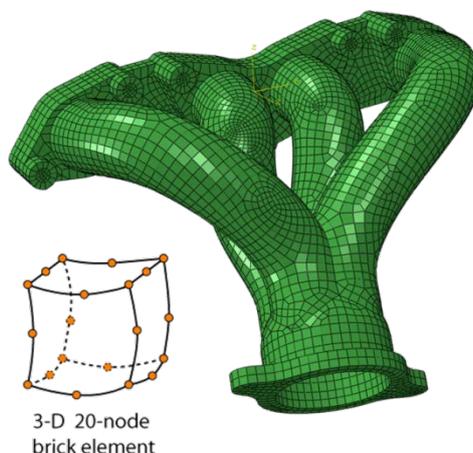


Figure 5: An exhaust manifold discretised (meshed) using 3-dimensional, quadratic brick elements, enabling the geometrical curvature to be captured

Coordinate Systems

Solution fields (such as displacement fields in solid mechanics) often require integrals of the following kind to be evaluated:

$$\int_{X_i}^{X_j} S_i(x)S_j(x)dx \text{ or } \int_{X_i}^{X_j} S_i^2(x)dx \quad (45)$$

Note that the superscript in Eqn.45 denotes a power now (and not an element label). These integrals can be simplified to make the integration procedures more efficient by deriving new shape functions defined relative to a local (element level) coordinate system.

For instance, we have so far derived our one-dimensional linear shape functions using a global coordinate system with its origin at O – Fig.6. These shape functions were:

$$S_i(x) = \left(\frac{X_j-x}{L}\right) \quad (46)$$

$$S_j(x) = \left(\frac{x-X_i}{L}\right) \quad (47)$$

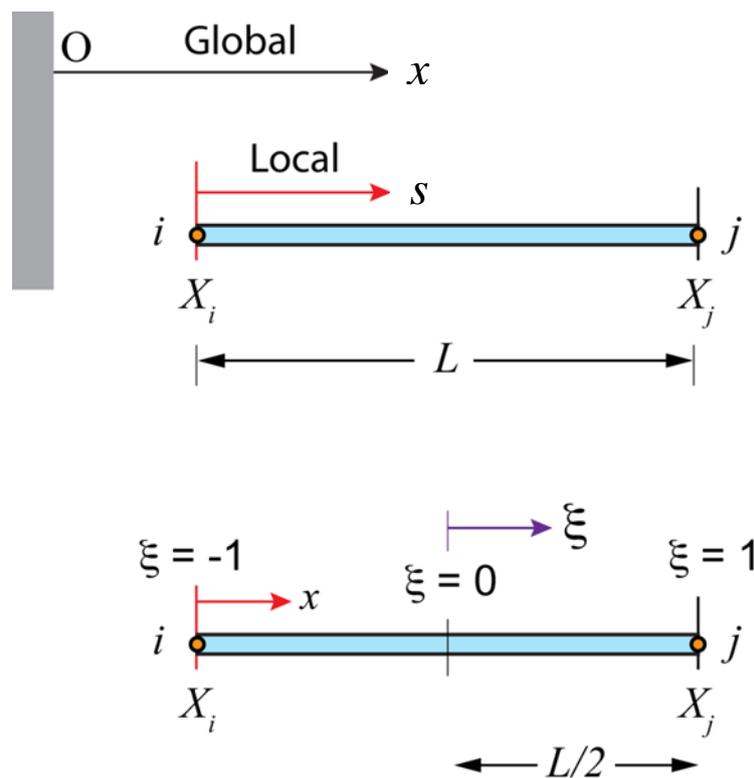


Figure 6: Representation of a single linear element in global, local and natural frames of reference

We wish now to derive the shape functions in the local coordinate reference frame with the origin at X_i - shown also in Fig.6. In the new reference frame the position along the element given previously by x is now given by $X_i + s$. We can rewrite the shape functions in Eqns.46 and 47 using the new reference frame by substitution such that:

$$S_i(s) = \frac{x_j - (x_i + s)}{L} = 1 - \frac{s}{L} \quad (48)$$

$$S_j(s) = \frac{(x_i + s) - x_i}{L} = \frac{s}{L} \quad (49)$$

and

$$\int_{x_i}^{x_j} S_i(x)S_j(x)dx \text{ or } \int_{x_i}^{x_j} S_i^2(x)dx \quad (50)$$

becomes

$$\int_0^L S_i(s)S_j(s)ds \text{ or } \int_0^L S_i^2(s)ds \quad (51)$$

where the limits of integration are easier to deal with. However, they can be simplified further by using a non-dimensional natural coordinate system. A natural coordinate system is shown in Fig.6. A coordinate at the centre of the element is specified. From this reference frame, the shape functions are:

$$S_i(\xi) = \frac{1}{2}(1 - \xi) \quad (52)$$

$$S_j(\xi) = \frac{1}{2}(1 + \xi) \quad (53)$$

To find these, we substitute x in Eqns.46 and 47 with $x = X_i + \frac{L}{2} + \xi$ and note that $L = 2$

$$S_i(\xi) = \frac{x_j - x}{L} = \frac{x_j - (x_i + \frac{L}{2} + \xi)}{L} = \frac{x_j}{L} - \frac{x_i}{L} - \frac{L}{2L} - \frac{\xi}{L} = \frac{1}{2} - \frac{\xi}{2} = \frac{1}{2}(1 - \xi) \quad (54)$$

$$S_j(\xi) = \frac{x - x_i}{L} = \frac{(x_i + \frac{L}{2} + \xi) - x_i}{L} = \frac{x_i}{L} + \frac{L}{2L} + \frac{\xi}{L} - \frac{x_i}{L} = \frac{1}{2} + \frac{\xi}{2} = \frac{1}{2}(1 + \xi) \quad (55)$$

Recall that the shape functions are equal to one at their respective nodes and 0 at the other nodes – i.e:

$$S_i(-1) = \frac{1}{2}(1 - \xi) = 1$$

$$S_i(1) = \frac{1}{2}(1 - \xi) = 0$$

$$S_j(-1) = \frac{1}{2}(1 + \xi) = 0$$

$$S_j(1) = \frac{1}{2}(1 + \xi) = 1$$

The integrals become:

$$\int_{-1}^1 S_i(\xi)S_j(\xi)d\xi \text{ or } \int_{-1}^1 S_i^2(\xi)d\xi \quad (56)$$

where the limits of integration are -1 and $+1$. Integrals of this type can be solved very efficiently using, for instance, the Gauss-Legendre method.