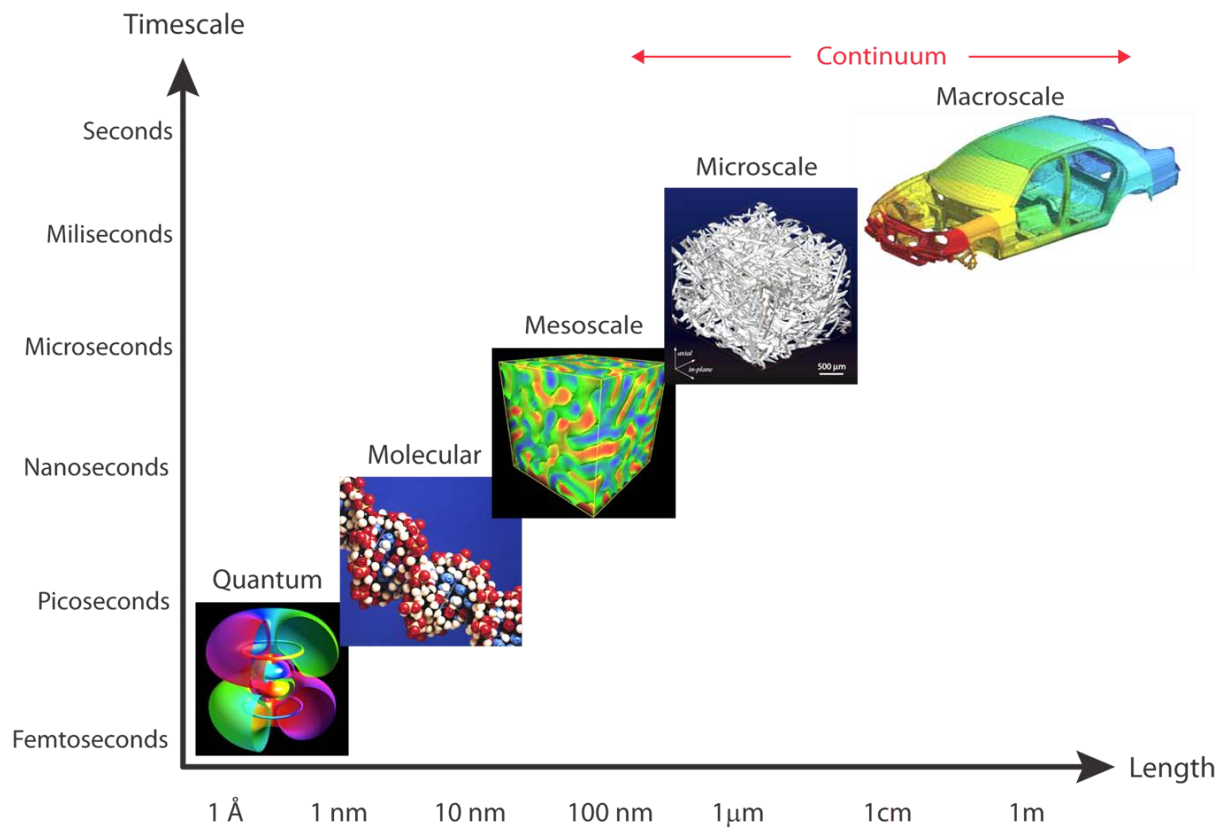University of Cambridge
Department of
Materials Science & Metallurgy

Introduction to the Finite Element Method (FEM)

Lecture 1
The Direct Stiffness Method and the Global StiffnessMatrix

Dr. J. Dean

Timescale

Seconds

Miliseconds

Microseconds

Nanoseconds

Picoseconds

Femtoseconds

Quantum

Molecular

Mesoscale

Microscale

Macroscale

Continuum

Length

1 Å   1 nm   10 nm   100 nm   1μm   1cm   1m

500 μm

axial
in-plane

Introduction

The finite element method (FEM) is a numerical technique for solving a wide range of complex physical phenomena, particularly those exhibiting geometrical and material non-linearities (such as those that are often encountered in the physical and engineering sciences). These problems can be structural in nature, thermal (or thermo-mechanical), electrical, magnetic, acoustic etc. plus any combination of. It is used most frequently to tackle problems that aren't readily amenable to analytical treatments.
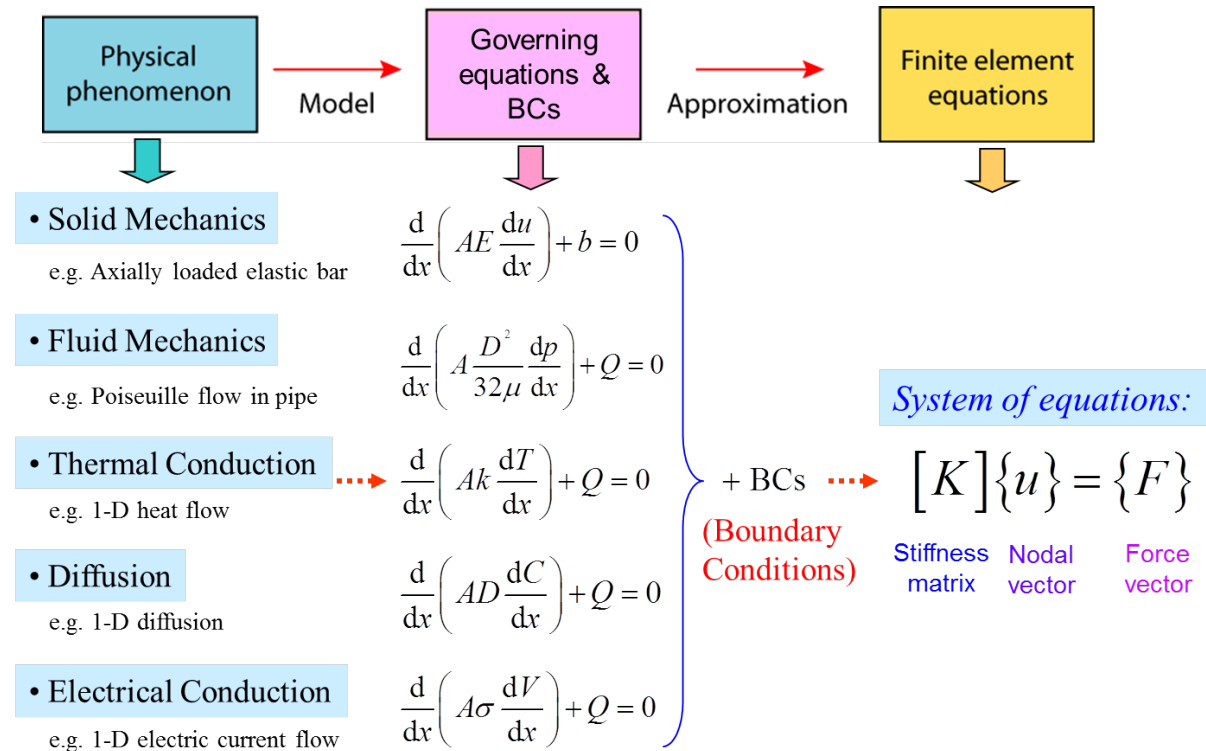

Figure 1: Governing equations for various physical phenomena

The premise is very simple; continuous domains (geometries) are decomposed into discrete, connected regions (or finite elements). An assembly of element-level equations is subsequently solved, in order to establish the response of the complete domain to a particular set of boundary conditions.

The Direct Stiffness Method and the Stiffness Matrix

There are several finite element methods. These are the *Direct Approach*, which is the simplest method for solving discrete problems in 1 and 2 dimensions; the *Weighted Residuals* method which uses the governing differential equations directly (e.g. the Galerkin method), and the *Variational Approach*, which uses the calculus of variation and the minimisation of potential energy (e.g. the Rayleigh-Ritz method).

We analyse the Direct Stiffness Method here, since it is a good starting point for understanding the finite element formulation. We consider first the simplest possible element – a 1-dimensional elastic spring which can accommodate only tensile and compressive forces. For the spring system shown in Fig.2, we accept the following conditions:

- *Condition of Compatibility – connected ends (nodes) of adjacent springs have the same displacements*
- *Condition of Static Equilibrium – the resultant force at each node is zero*
- *Constitutive Relation – that describes how the material (spring) responds to the applied loads*
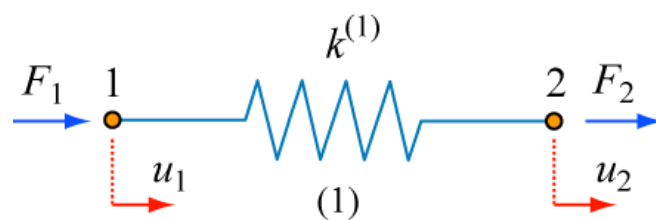

Figure 6: Model spring system

The constitutive relation can be obtained from the governing equation for an elastic bar loaded axially along its length:

$$\frac{d}{du}\left(AE\frac{\Delta l}{l_0}\right) + k = 0 \tag{1}$$

$$\frac{\Delta l}{l_0} = \varepsilon \tag{2}$$

$$\frac{d}{du}(AE\varepsilon) + k = 0 \tag{3}$$
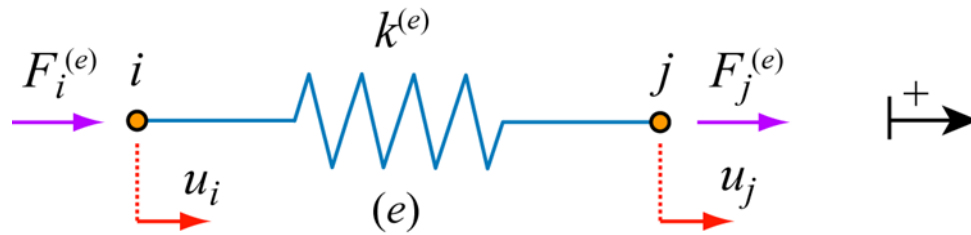
$$\frac{d}{du}(A\sigma) + k = 0 \tag{4}$$

$$\frac{dF}{du} + k = 0 \tag{5}$$

$$\frac{dF}{du} = -k \tag{6}$$

$$dF = -kdu \tag{7}$$

The spring stiffness equation relates the nodal displacements to the applied forces via the spring (element) stiffness. From here on in we use the scalar version of Eqn.7.

From inspection, we can see that there are two degrees of freedom in this model, u_i and u_j. We can write the force equilibrium equations:

$$k^{(e)}u_i - k^{(e)}u_j = F_i^{(e)} \tag{8}$$

$$-k^{(e)}u_i + k^{(e)}u_j = F_j^{(e)} \tag{9}$$

In matrix form

$$\begin{bmatrix} k^e & -k^e \\ -k^e & k^e \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} F_i^{(e)} \\ F_j^{(e)} \end{Bmatrix} \tag{10}$$

The order of the matrix is [2×2] because there are 2 degrees of freedom. Note also that the matrix is symmetrical. The 'element' stiffness relation is:

$$[K^{(e)}]\{u^{(e)}\} = \{F^{(e)}\} \tag{11}$$

Where $K^{(e)}$ is the element stiffness matrix, $u^{(e)}$ the nodal displacement vector and $F^{(e)}$ the nodal force vector. (The element stiffness relation is important because it can be used as a building block for more complex systems. An example of this is provided later.)

Derivation of a Global Stiffness Matrix

For a more complex spring system, a 'global' stiffness matrix is required – i.e. one that describes the behaviour of the complete system, and not just the individual springs.
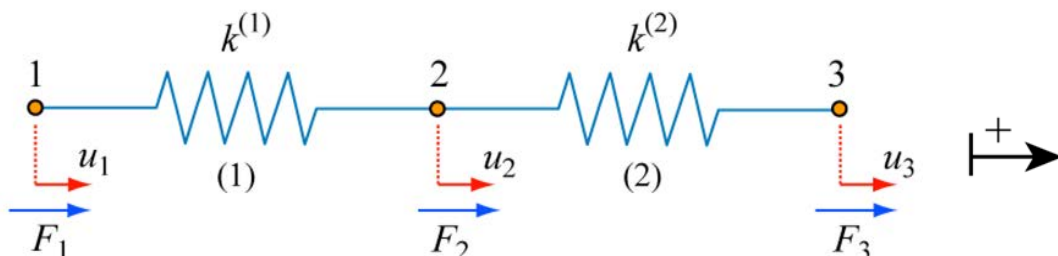
From inspection, we can see that there are two springs (elements) and three degrees of freedom in this model, u_1, u_2 and u_3. As with the single spring model above, we can write the force equilibrium equations:

$$k^1 u_1 - k^1 u_2 = F_1 \tag{12}$$

$$-k^1 u_1 + (k^1 + k^2)u_2 - k^2 u_3 = F_2 \tag{13}$$

$$k^2 u_3 - k^2 u_2 = F_3 \tag{14}$$

In matrix form

$$\begin{bmatrix} k^1 & -k^1 & 0 \\ -k^1 & k^1 + k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \tag{15}$$

The 'global' stiffness relation is written in Eqn.16, which we distinguish from the 'element' stiffness relation in Eqn.11.

$$[K]\{u\} = \{F\} \tag{16}$$

Note the shared k^1 and k^2 at $k22$ because of the compatibility condition at u_2. We return to this important feature later on.

Assembling the Global Stiffness Matrix from the Element Stiffness Matrices

Although it isn't apparent for the simple two-spring model above, generating the global stiffness matrix (directly) for a complex system of springs is impractical. A more efficient method involves the assembly of the individual element stiffness matrices. For instance, if you take the 2-element spring system shown,
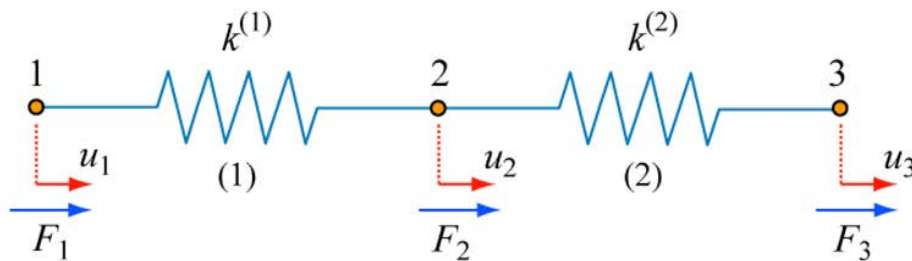
split it into its component parts in the following way
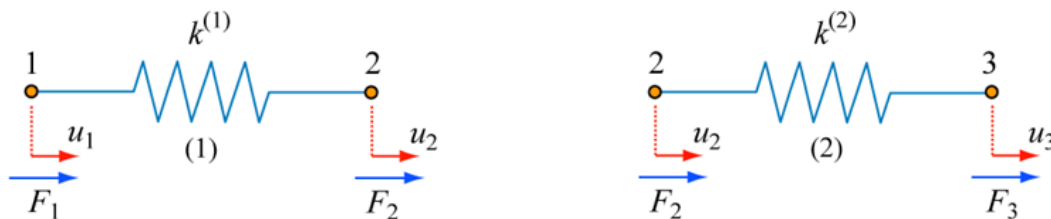
and derive the force equilibrium equations

$$k^1 u_1 - k^1 u_2 = F_1 \tag{17}$$

$$k^1 u_2 - k^1 u_1 = k^2 u_2 - k^2 u_3 = F_2 \tag{18}$$

$$k^2 u_3 - k^2 u_2 = F_3 \tag{19}$$

then the individual element stiffness matrices are:

$$\begin{bmatrix} k^1 & -k^1 \\ -k^1 & k^1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \text{ and } \begin{bmatrix} k^2 & -k^2 \\ -k^2 & k^2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \tag{20}$$

such that the global stiffness matrix is the same as that derived directly in Eqn.15:

$$\begin{bmatrix} k^1 & -k^1 & 0 \\ -k^1 & k^1 + k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \tag{21}$$

(Note that, to create the global stiffness matrix by assembling the element stiffness matrices, k22 is given by the sum of the direct stiffnesses acting on node 2 – which is the compatibility criterion. Note also that the indirect cells (k_{ij}) are either zero (no load transfer between nodes i and j), or negative to indicate a reaction force.)

For this simple case the benefits of assembling the element stiffness matrices (as opposed to deriving the global stiffness matrix directly) aren't immediately obvious. We consider therefore the following (complex) system which contains 5 springs (elements) and 5 degrees of freedom (problems of practical interest can have tens or hundreds of thousands of degrees of freedom (and more!)). Since there are 5 degrees of freedom we know the matrix order is 5×5. We also know that it's symmetrical, so it takes the form shown below:
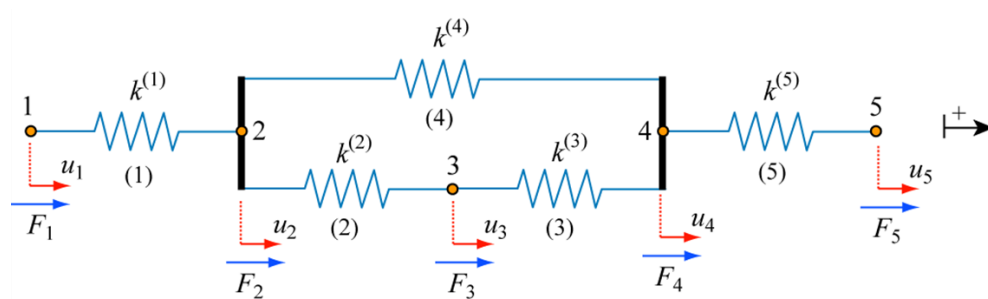
$$\boxed{[K]\{u\} = \{F\}}$$
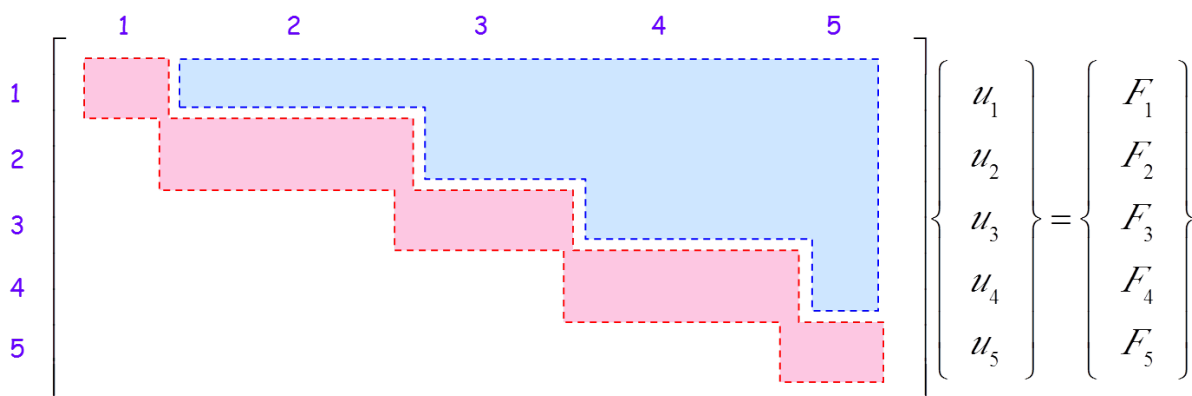
We want to populate the cells to generate the global stiffness matrix. From our observation of simpler systems, e.g. the two spring system above, the following rules emerge:

- The term in location ii consists of the sum of the *direct stiffnesses* of all the elements meeting at node i
- The term in location ij consists of the sum of the indirect stiffnesses relating to nodes i and j of all the elements joining node i to j
- Add a negative for reaction terms $(-k_{ij})$
- Add a zero for node combinations that don't interact

By following these rules, we can generate the global stiffness matrix:

$$
\begin{array}{ccccc}
 & 1 & 2 & 3 & 4 & 5
\end{array}
$$

$$
\begin{array}{c}
1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{bmatrix}
k^{(1)} & -k^{(1)} & 0 & 0 & 0 \\
-k^{(1)} & k^{(1)}+k^{(2)}+k^{(4)} & -k^{(2)} & -k^{(4)} & 0 \\
0 & -k^{(2)} & k^{(2)}+k^{(3)} & -k^{(3)} & 0 \\
0 & -k^{(4)} & -k^{(3)} & k^{(3)}+k^{(4)}+k^{(5)} & -k^{(5)} \\
0 & 0 & 0 & -k^{(5)} & k^{(5)}
\end{bmatrix}
\begin{Bmatrix}
u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5
\end{Bmatrix}
=
\begin{Bmatrix}
F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5
\end{Bmatrix}
$$

This type of assembly process is handled automatically by commercial FEM codes

Solving for (u)

The unknowns (degrees of freedom) in the spring systems presented are the displacements u_{ij}. Our global system of equations takes the following form:

$$[K]\{u\} = \{F\}$$

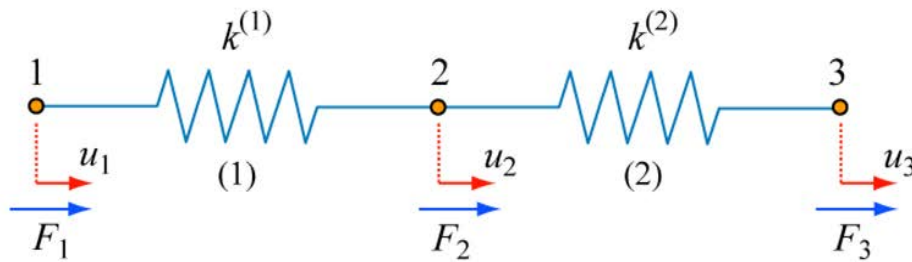Global Stiffness Matrix

Nodal displacements (unknowns)

Nodal forces

To find $\{u\}$ solve

$$\{u\} = \{F\}[K]^{-1} \tag{22}$$

Recall that $[k][k]^{-1} = I = \text{Identitiy Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Recall also that, in order for a matrix to have an inverse, its determinant must be non-zero. If the determinant is zero, the matrix is said to be singular and no unique solution for Eqn.22 exists. For instance, consider once more the following spring system:

We know that the global stiffness matrix takes the following form

$$\begin{bmatrix} k^1 & -k^1 & 0 \\ -k^1 & k^1 + k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \qquad (23)$$

The determinant of $[K]$ can be found from:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = (aei + bfg + cdh) - (ceg + bdi + afh) \qquad (24)$$

Such that:

$$(k^1(k^1 + k^2)k^2 + 0 + 0) - \left(0 + (-k^1-k^1k^2) + (k^1-k^2 - k^2)\right) \qquad (25)$$

$$\det[K] = \left(k^{1^2}k^2 + k^1k^{2^2}\right) - \left(k^{1^2}k^2 + k^1k^{2^2}\right) = 0 \qquad (26)$$

Since the determinant of $[K]$ is zero it is not invertible, but singular. There are no unique solutions and $\{u\}$ cannot be found.

Enforcing Boundary Conditions

By enforcing boundary conditions, such as those depicted in the system below, $[K]$ becomes invertible (non-singular) and we can solve for the reaction force F_1 and the unknown displacements $\{u_2\}$ and $\{u_3\}$, for known (applied) F_2 and F_3.
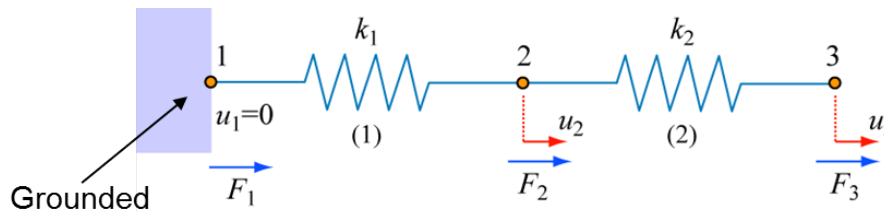
No BC:

$$\begin{bmatrix} k^{(1)} & -k^{(1)} & 0 \\ -k^{(1)} & k^{(1)}+k^{(2)} & -k^{(2)} \\ 0 & -k^{(2)} & k^{(2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \quad \longrightarrow \quad \det[K]=0$$

Enforce BC:
$u_1 = 0$

$$\begin{bmatrix} k^{(1)} & -k^{(1)} & 0 \\ -k^{(1)} & k^{(1)}+k^{(2)} & -k^{(2)} \\ 0 & -k^{(2)} & k^{(2)} \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \quad \longrightarrow \quad \det[K] \neq 0$$

$$[K] = \begin{bmatrix} k^1+k^2 & -k^2 \\ -k^2 & k^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{27}$$

$$det[K] = ab - cd \tag{28}$$

$$det[K] = (k^1+k^2)k^2 - k^{2^2} = k^1 k^2 \neq 0 \tag{29}$$

Unique solutions for F_1, $\{u_2\}$ and $\{u_3\}$ can now be found

$-k^1 u_2 = F_1$

$(k^1+k^2)u_2 - k^2 u_3 = F_2 = k^1 u_2 + k^2 u_2 - k^2 u_3$

$-k^2 u_2 + k^2 u_3 = F_3$

Unique solutions:

$$-k^{(1)}u_2 = F_1$$

$$u_2 = \frac{1}{k^{(1)}}(F_2 + F_3)$$

$$u_3 = \frac{F_2}{k^{(1)}} + \left(\frac{1}{k^{(1)}} + \frac{1}{k^{(2)}} \right)F_3$$

In this instance we solved three equations for three unknowns. In problems of practical interest the order of $[K]$ is often very large and we can have thousands of unknowns. It then becomes impractical to solve for $\{u\}$ by inverting the global stiffness matrix. We can instead use Gauss elimination which is more suitable for solving systems of linear equations with thousands of unknowns.

Gauss Elimination

We have a system of equations

$$x - 3y + z = 4 \qquad (30)$$

$$2x - 8y + 8z = -2 \qquad (31)$$

$$-6x + 3y - 15z = 9 \qquad (32)$$

when expressed in augmented matrix form

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{array}\right] \qquad (33)$$

We wish to create a matrix of the following form

$$\left[\begin{array}{ccc|c} 11 & 12 & 13 & 1 \\ 0 & 22 & 23 & 2 \\ 0 & 0 & 33 & 3 \end{array}\right] \qquad (34)$$

Where the terms below the direct terms are zero

We need to eliminate some of the unknowns by solving the system of simultaneous equations

To eliminate x from row 2 (where R denotes the row)

$$-2(R1) + R2 \qquad (35)$$

$$-2(x - 3y + z) + (2x - 8y + 8z) = -10 \qquad (36)$$

$$-2y + 6z = -10 \qquad (37)$$

So that

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ -6 & 3 & -15 & 9 \end{array}\right] \qquad (38)$$

To eliminate x from row 3

$$6(R1) + R3 \qquad (39)$$

$$6(x - 3y + z) + (-6x + 3y - 15z) = 33 \qquad (40)$$

$$-15y - 9z = 33 \qquad (41)$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ 0 & -15 & -9 & 33 \end{array}\right] \qquad (42)$$

To eliminate y from row 2

R2/2

$$-y + 3z = -5 \tag{43}$$

$$\begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & -15 & -9 & 33 \end{bmatrix} \tag{44}$$

To eliminate y from row 3

R3/3

$$-5y - 3z = -11 \tag{45}$$

$$\begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & -5 & -3 & 11 \end{bmatrix} \tag{46}$$

And then

-5(R2) + R3 (47)

$$-5(-y + 3z) + (-5y - 3z) = 36 \tag{48}$$

$$-18z = 36 \tag{49}$$

$$\begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & -18 & 36 \end{bmatrix} \tag{50}$$

$$-2 = z \tag{51}$$

Substituting z = -2 back in to R2 gives y = -1

Substituting y = -1 and z = -2 back in to R1 gives x = 3

This process of progressively solving for the unknowns is called *back substitution*.

Consider a wall mounted bracket loaded uniformly along its length as in Fig.2
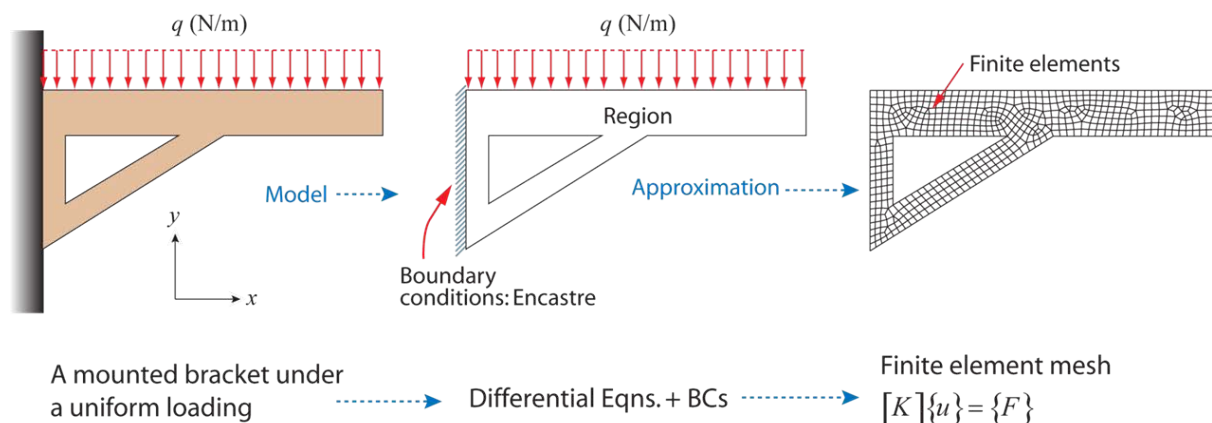

Figure 2: Wall mounted bracket

The geometry is defined for us and is (relatively) complex. The boundary conditions are also defined and are:

- · A uniform force per unit length along the upper edge
- · Fixed *x* and *y* displacements along the clamped edge

It is apparent that the bracket will respond mechanically under the action of the applied load and a system of internal stresses will develop (to balance the applied load). To calculate the stresses that develop we must first discretise the domain, assemble the global stiffness matrix $[K]$, and then determine the nodal displacements $\{u\}$ and resultant forces $\{F\}$ using some iterative numerical technique (Gauss elimination, for instance). It is then a relatively trivial exercise to compute the stresses from the displacements (particularly for systems that remain elastic).